ON THE SOLUTION OF A BOUNDARY VALUE PROBLEM ON THE SLOSHING OF A LIQUID IN CONICAL CAVITIES

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The problem of determining the dynamic characteristics of a sloshing fluid partially filling a moving cylindrical cavity was solved by G.S. Narimanov, D.E. Okhotsimskii, B.I. Rabinovich and N.N. Moiseev. The values of the free vibration frequencies of a shallow fluid in a conical bottom with large apex angles were obtained in [1] by a variational method.

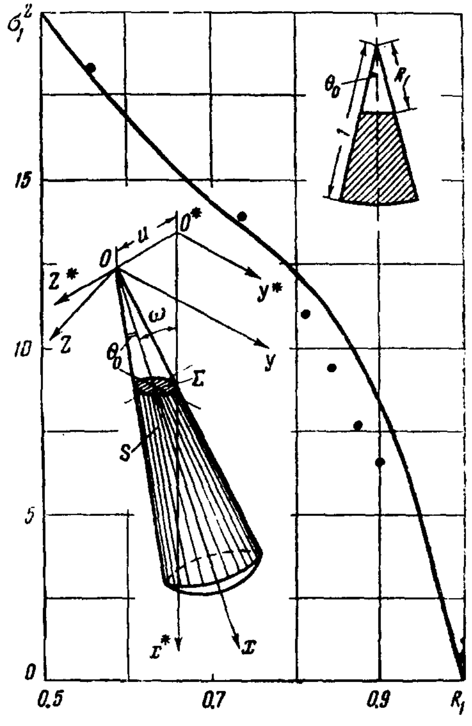

Fig. 1

An analytical solution of the problem of fluid sloshing in a moving cone with a small apex angle is given below, and not only the frequencies but also the apparent mass of the fluid in cavities with any coning angle are determined by a variational method.

Let us represent the displacement potential of an ideal incompressible fluid in the form of the sum

$$\Phi(x, y, z, t) = zu(t) + \Psi(x, y, z)\omega(t) + \sum_{n=1}^{\infty} \varphi_n(x, y, z)\, r_n(t) \qquad (1)$$

Here u is the displacement, ω the rotation of the cavity and r_n a generalized coordinate of fluid sloshing (Fig. 1).

From the condition of impermeability on the wetted surface S and the constancy of the pressure on the free surface Σ, we have the following boundary conditions for Ψ and φ_n:

$$\Delta\Psi=0, \quad \frac{\partial\Psi}{\partial\nu}=0 \text{ on } \Sigma \qquad \frac{\partial\Psi}{\partial\nu}=z\nu_x-x\nu_z \text{ on } S \tag{2}$$

$$\Delta\varphi=0, \quad \frac{\partial\varphi_n}{\partial\nu}=0 \text{ on } S \quad \frac{\partial\varphi_n}{\partial\nu}=\frac{\varphi_n}{\varphi_n(x^\circ,0,z^\circ)} \text{ on } \Sigma \tag{3}$$

The equations of the disturbed motion of a cavity partially filled with fluid in the x^*z^* plane have the form

$$\begin{gathered}(\mu^\circ+\mu)u^{\cdot\cdot}+\sum_{n=1}^{\infty}\lambda_n r_n^{\cdot\cdot}=P\\ (J^\circ+J)\omega^{\cdot\cdot}+\sum_{n=1}^{\infty}\lambda_{0n}r_n^{\cdot\cdot}=M\\ \mu_n(r_n^{\cdot\cdot}+\sigma_n^2 r_n)+\lambda_n u^{\cdot\cdot}+\lambda_{0n}\omega^{\cdot\cdot}=0\\ (n=1,2,\ldots)\end{gathered} \tag{4}$$

Here $\mu^\circ+\mu$ is the mass of the body plus the fluid, J° is the moment of inertia of the solid body relative to the y-axis and the apparent masses $\lambda_n, \lambda_{0n}, \mu_n, J$ and the fluid sloshing frequencies σ_n^2 are determined by Formulas

$$\lambda_n=\int_S\varphi_n\nu_z dS=\int_\Sigma z\frac{\partial\varphi_n}{\partial\nu}dS, \quad \lambda_{0n}=\int_S\varphi_n(z\nu_x-x\nu_z)\,dS=\int_\Sigma\Psi\frac{\partial\varphi_n}{\partial\nu}dS \tag{5}$$

$$\mu_n=\int_\Sigma\varphi_n\frac{\partial\varphi_n}{\partial\nu}dS, \qquad J=\int_S\Psi(z\nu_x-x\nu_z)\,dS, \qquad \sigma_n^2=\frac{1}{\varphi_n(x^\circ,0,z^\circ)}$$

It is assumed that a unit mass-force vector acts on a fluid with unit density. The free surface profile Σ intersects the xz plane at the point $(x^\circ, 0, z^\circ)$.

By virtue of (2) and (3) the boundary value problem for the cone in the spherical R, θ, η coordinate system (Fig. 1) is written in the form

$$\Delta\varphi_n=0, \quad \left[\frac{\partial\varphi_n}{\partial R}\right]_{R=1}=\left[\frac{\partial\varphi_n}{\partial\theta}\right]_{\theta=\theta_0}=0, \quad \left[\frac{\partial\varphi_n}{\partial R}\right]_{R=R_1}=-\sigma_n^2[\varphi_n]_{R=R_1} \tag{6}$$

$$\Delta\Psi=0, \quad \left[\frac{\partial\Psi}{\partial R}\right]_{R=1,R_1}=0, \quad \left[\frac{\partial\Psi}{\partial\theta}\right]_{\theta=\theta_0}=-R^2\sin\eta \tag{7}$$

The boundary condition of constancy of the pressure on the disturbed free surface Σ is here transferred to the surface of the spherical segment $R=R_1$, $0\leqslant\theta<\theta_0$, which is possible as long as the height of the latter is commensurate with the height of the wave.

Using separation of variables, we obtain the solution of the boundary value problem (6)

$$\varphi_n=X_n(R)\,Y_n(\theta)\sin\eta, \quad Y_n=\frac{P_\nu(\theta)}{P_\nu(\theta_0)}, \qquad X_n=\frac{R_1^{\nu+2}\,[(\nu+1)R^{2\nu+1}+\nu]}{R^{\nu+1}\nu(\nu+1)(R_1^{2\nu+1}-1)} \tag{8}$$

Here ν is the n-th root of Equation $dP_\nu(\theta_0)/d\theta=0$, and $P_\nu(\theta)$ is the first associated Legendre function of the first kind of ν-th order.

The function Ψ can be represented as

$$\Psi = -R^2\theta\sin\eta + \sum_{n=1}^{\infty} A_n(2R_1X_n + 2X_n^*)\,Y_n\sin\eta + \sum_{n=1}^{\infty} Q_n(R)\,Y_n\sin\eta$$

which, after certain transformations may be written

$$\Psi = \sum_{n=1}^{\infty} \frac{\sin\theta_0}{(\nu+3)(\nu-2)N_n^2}(2R_1X_n + 2X_n^* - R^2)\,Y_n\sin\eta \tag{9}$$

$$N_n^2 = \frac{\gamma_n\sin\theta_0}{(2\nu+1)},\qquad \gamma_n = -\frac{d}{d\nu}[Y_n'(\theta_0)],\quad X_n^* = -\frac{(\nu+1)R^{2\nu+1} + \nu R_1^{2\nu+1}}{R^{\nu+1}\nu(\nu+1)(R_1^{2\nu+1}-1)}$$

The coefficients (5) of the system of equations of motion (4) of a conical cavity (Fig. 1) with fluid take the form

$$\lambda_n = \frac{\pi R_1^3\sin\theta_0\cos\theta_0}{(\nu+2)(\nu-1)}\left[\frac{(\nu+1)R_1^{2\nu+1}+\nu}{\nu(\nu+1)(R_1^{2\nu+1}-1)} - 1\right]$$

$$\lambda_{0n} = \frac{\pi R_1^4\sin\theta_0}{(\nu+3)(\nu-2)}\left[1 - 2\frac{(\nu+1)R_1^{2\nu+1} - (2\nu+1)R_1^{\nu-2}+\nu}{\nu(\nu+1)(R_1^{2\nu+1}-1)}\right]$$

$$\mu_n = \frac{\pi R_1^3N_n^2[(\nu+1)R_1^{2\nu+1}+\nu]}{\nu(\nu+1)(R_1^{2\nu+1}-1)},\qquad \sigma_n^2 = \frac{\nu(\nu+1)(1-R_1^{2\nu+1})}{R_1[(\nu+1)R_1^{2\nu+1}+\nu]} \tag{10}$$

$$J = \sum_{n=1}^{\infty}\frac{\pi\sin^2\theta_0}{(\nu+3)(\nu-2)N_n^2}\left\{\left[\frac{1}{5} - \frac{2}{(\nu+3)(\nu-2)}\right](1-R_1^5) - \right.$$
$$\left. -4\,\frac{(\nu+1)+\nu R_1^{2\nu+1} + R_1^5[(\nu+1)R_1^{2\nu+1}+\nu] - 2(2\nu+1)R_1^{\nu+3}}{(\nu+3)(\nu-2)\nu(\nu+1)(R_1^{2\nu+1}-1)}\right\}$$

The coefficients for a cone with a vertex turned downward may be obtained analogously. As an example, let us present the expression for the coefficients for a pointed, untruncated conical bottom (Fig.2)

$$\lambda_n = \frac{\pi\sin\theta_0\cos\theta_0}{\nu(\nu+2)},\qquad \lambda_{0n} = -\frac{\pi\sin\theta_0}{\nu(\nu+3)},\qquad \sigma_n^2 = \nu$$

$$\mu_n = \frac{\pi\gamma_n\sin\theta_0}{\nu(2\nu+1)},\qquad J = \sum_{n=1}^{\infty}\frac{\pi\sin\theta_0(2\nu+1)(\nu+5)}{5\gamma_n\nu(\nu+3)^2} \tag{11}$$

Let us present the values of the first root $\nu(n=1)$ of Equation $Y_1'(\theta_0) = 0$ and the corresponding values of γ_1 for certain values of the cone half apex angle θ_0

$\theta_0 =$	1°	3°	7°	11°	15°	20°	25°	30°
$\nu =$	105.0	34.67	14.59	9.128	6.584	4.843	3.806	3.120
$\gamma_1 =$	1.298	1.300	1.302	1.307	1.313	1.325	1.341	1.360

Fig. 1 shows the good agreement between the experimental, nondimensional values of the frequency of the first mode of the fluid sloshing σ_1^2 in a cone ($\theta_0 = 11°$) and the theoretical values (10) as a function of R_1

The obtained solutions of boundary value problems (6) and (7) start to yield a significant discrepancy from practice for blunt cones. This is explained by the boundary condition on the free surface not being satisfied exactly.

Let us reduce the solution of boundary value problems (2) and (3) to the determination of the extremum of the functionals. Using a variational method [1], we obtain

$$V_1 = \frac{1}{2}\int\limits_Q (\mathrm{grad}\,\varphi)^2\, dQ - \frac{1}{2}\sigma^2\int\limits_\Sigma \varphi^2 dS \qquad (12)$$

$$V_2 = \frac{1}{2}\int\limits_Q (\mathrm{grad}\,F)^2\, dQ - \int\limits_\Sigma Fz\nu_x\, dS - \int\limits_S 2Fz\nu_x dS, \qquad F = \Psi + xz \qquad (13)$$

Following the Ritz method, let us seek the functions φ and F as sequences of linear combinations of the coordinate functions

$$\varphi = a_1\gamma_1 + \ldots + a_k\gamma_k,$$
$$F = c_1\zeta_1 + \ldots + c_k\gamma_k \qquad (14)$$

A system of linearly independent functions, complete with respect to the energy, should be taken as coordinate functions for the convergence of the Ritz method.

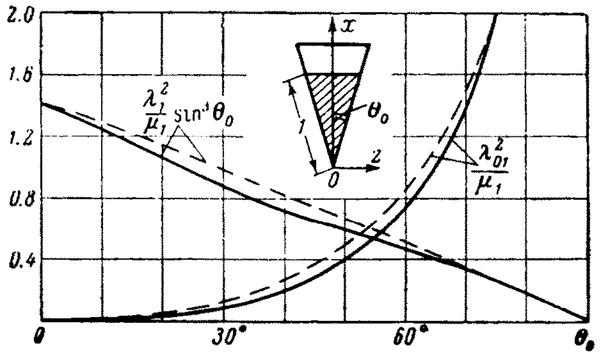

Fig. 2

Substituting the finite sums (14) into the functionals (12) and (13), and equating the partial derivatives of the functionals with respect to the undetermined coefficients a_m and c_m to zero, we obtain a homogeneous linear system of algebraic equations for a_m and an inhomogeneous system for c_m

$$(\mathbf{A} - \sigma^2\mathbf{B})\cdot\mathbf{a} = 0, \qquad \mathbf{D}\cdot\mathbf{c} = \boldsymbol{\eta} \qquad (15)$$

Here A, B and D are symmetric square matrices of rank k composed from the coefficients α_{mp}, β_{mp}, δ_{mp}, respectively, and the quantities $\mathbf{a}$, $\mathbf{c}$ and η are k-dimensional vectors with components a_m, c_m and η_m .

For the harmonic functions γ_m and ζ_m

$$\alpha_{mp} = \alpha_{pm} = \int\limits_{S+\Sigma} \gamma_m \frac{\partial\gamma_p}{\partial\nu} dS, \quad \beta_{mp} = \beta_{pm} = \int\limits_\Sigma \gamma_m\gamma_p dS \qquad (16)$$

$$\delta_{mp} = \delta_{pm} = \int\limits_{S+\Sigma} \zeta_m \frac{\partial\zeta_p}{\partial\nu} dS, \qquad \eta_m = \int\limits_\Sigma \zeta_m z\nu_x dS + 2\int\limits_S \zeta_m z\nu_x dS$$

As the number of terms k of the sum (14) increases, the n-th eigenvalue σ_n^2 of the matrix of the first system of Equations (15) converges to values of the partial frequency of the n-th mode of the fluid shloshing in the cavity, and the function φ_n (14) determined by the n-th eigenvector of the matrix (15) approximates the wave shape of the n-th mode .

The apparent masses of fluid sloshing in the cavity will be determined by Formulas

$$\lambda_n = \sum_{m=1}^{k} a_m\lambda_m^*, \quad \mu_n = (\mathbf{A}\cdot\mathbf{a}, \mathbf{a}) = \sigma_n^2(\mathbf{B}\cdot\mathbf{a}, \mathbf{a})$$

$$\lambda_{0n} = \sum_{m=1}^{k} a_m\lambda_{0m}^*, \; J = J_0 + \sum_{m=1}^{k} c_m(\eta_m - 2\tau_m)$$

where

$$\lambda_m^* = \int\limits_S \gamma_m v_z dS, \quad \lambda_{0m}^* = \int\limits_S \gamma_m (z v_x - x v_z)\, dS$$

$$J_\bullet = \int\limits_Q (x^2 + z^2)\, dQ, \quad \tau_m = \int\limits_{S+\Sigma} \zeta_m (z v_x + x v_z)\, dS$$

We will use the constructed complete system of functions (8) and (9) as the system of coordinate functions γ_m and ζ_m for the Ritz method.

Fig. 3 illustrates the character of the convergence of the apparent masses and frequencies for successive approximations of the Ritz method for the $\theta_0 = 11°$ case. Plotted along the horizontal axis is the number k of terms of sum in the Ritz method. Plotted along the vertical is the ratio of any of the coefficients of the k-th approximation to the corresponding coefficient taken as exact. The number $k = 0$ corresponds to Formulas (10). The nature of the convergence of the frequency σ_1^2 indicates the effectiveness of the selection of the basis system of functions. Fig. 2 shows the ratios of coefficients λ_1^2/μ_1 and λ_{01}^2/μ_1 for conical bottoms. These coefficients are independent of the method of selecting the normalization, and are determined both by the basic system of functions constructed above (dashes) and by cylindrical functions which are the exact solutions of the boundary value problems (2) and (3) for a cylinder circumscribed around the cone (solid line). The values σ_1^2 determined by both methods, agree (curve 1 in Fig. 4). Frequencies determined by Formulas (11) (curve 5), by Formula $\sigma_1^2 \sin\theta_0 = 1.1459 \cot\theta_0$ (curve 3) [1] and also the frequencies in a cylinder circumscribed around the cone (curve 6) are compared with them. Curve 2 is determined by the variational method for a cone with vertex up and curve 4 by Formula (10). The experimental points were obtained by Mikishev [2]. All the work associated with the programing and computations on the electronic computer (EVTsM) was performed by E.S. Mironova.

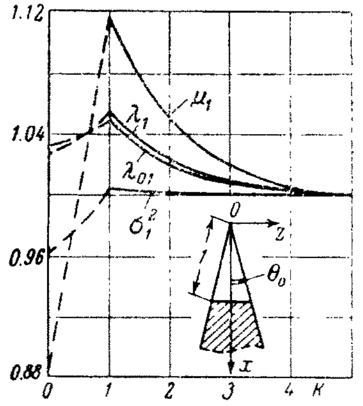

Fig. 3

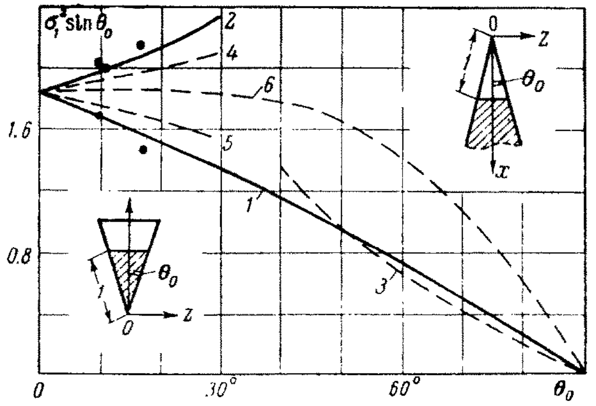

Fig. 4

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